§5.3 Pions as Goldstone Bosons

Example in particle physics:
approximate symmetry of strong interactions
$\longrightarrow$ chiral $SU(2) \times SU(2)$

2 quark fields, u and d, with very small mass:

$$\mathcal{L} = -\bar{u}\gamma^\mu D_\mu u - \bar{d}\gamma^\mu D_\mu d - \dots, \qquad (1)$$

where $D_\mu = \partial_\mu - iA_\mu$ and "..." are independent of u and d
(take the limit with vanishing masses)
$\longrightarrow$ invariant under

$$\begin{pmatrix} u \\ d \end{pmatrix} \longmapsto \exp\left(i\vec{\theta}^V \cdot \vec{t} + i\gamma_5 \vec{\theta}^A \cdot \vec{t}\right)\begin{pmatrix} u \\ d \end{pmatrix}$$

where $\vec{t}$ is three-vector of isospin matrices

$$t_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad t_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\vec{\theta}^V$ and $\vec{\theta}^A$ are independent real 3-vectors.
To see this, note

$$\overline{\gamma_5 \psi}\,\gamma^\mu = -\bar{\psi}\gamma_5\gamma^\mu = +\bar{\psi}\gamma^\mu\gamma^5 \quad \text{and} \quad \gamma^{5\,2} = \mathbb{1}$$

another representation:

$$\vec{t}_L = \frac{1}{2}(1+\gamma_5)\vec{t}, \qquad \vec{t}_R = \frac{1}{2}(1-\gamma_5)\vec{t}$$

acting on left-handed part   acting on right-handed part

with commutation relations

$$[t_{Li}, t_{Lj}] = i\varepsilon_{ijk} t_{Lk}, \qquad SU(2)_1$$

$$[t_{Ri}, t_{Rj}] = i\varepsilon_{ijk} t_{Rk}, \qquad SU(2)_2$$

$$[t_{Li}, t_{Rj}] = 0. \qquad \text{independent}$$

$\longrightarrow SU(2) \times SU(2)$

another subgroup:

ordinary isospin trfs. with $\vec{\theta}^A = 0$ and generators $\vec{t} = \vec{t}_L + \vec{t}_R$

$\longrightarrow SU(2) \times SU(2)$ may be written in terms of $\vec{t}$ and $\vec{x} = \vec{t}_L - \vec{t}_R = \gamma_5 \vec{t}$

with commutation relations

$$[t_i, t_j] = i\varepsilon_{ijk} t_k,$$

$$[t_i, x_j] = i\varepsilon_{ijk} x_k,$$

$$[x_i, x_j] = i\varepsilon_{ijk} t_k$$

$SU(2) \times SU(2)$ is spontaneously broken

$$\begin{array}{l} \mathcal{L}_{UV} \quad SU(2) \times SU(2) \\ \Big\downarrow \text{RG} \\ \mathcal{L}_{IR} \quad SU(2) \text{ with generator } \vec{t} \text{ preserved} \end{array}$$

$\longrightarrow$ Noether's method gives conserved currents

$\vec{V}^\mu = i\bar{q}\gamma^\mu \vec{t} q$ (vector),

$\vec{A}^\mu = i\bar{q}\gamma^\mu\gamma_5\vec{t} q$ (axial-vector)

$$\partial_\mu \vec{V}^\mu = 0 = \partial_\mu \vec{A}^\mu = 0$$

and where q is the quark doublet

$$q = \begin{pmatrix} u \\ d \end{pmatrix}$$

$\longrightarrow$ associated charges:

$$\vec{T} = \int d^3x\, \vec{V}^0,$$

$$\vec{X} = \int d^3x\, \vec{A}^0$$

satisfy the same commutation relations as $\vec{t}$ and $\vec{x}$:

$$[T_i, T_j] = i\varepsilon_{ijk} T_k, \quad [T_i, X_j] = i\varepsilon_{ijk} X_k,$$

$$[X_i, X_j] = i\varepsilon_{ijk} T_k$$

with action on quark fields given by

$$[\vec{T}, q] = -\vec{t}\, q,$$

$$[\vec{X}, q] = -\vec{x}\, q$$

Chiral SU(2)-symmetry generated by $\vec{X}$ is spontaneously broken in QCD

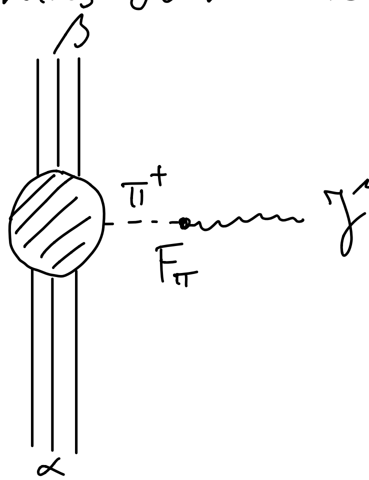
⟶ approximately massless Goldstone bosons with negative parity, zero spin, unit isospin, and zero baryon number (quantum numbers of $\vec{x}$)

⟶ theoretical reason for existence of pions!

Pions are emitted in "weak interactions":

$$\mathcal{L}_{wk} = -i\,\frac{G_{wk}}{\sqrt{2}}\left(V_+^\lambda + A_+^\lambda\right)\sum_\ell \bar{\ell}\,\gamma_\lambda(1+\gamma_5)\,\nu_\ell + \text{h.c.}$$

where ℓ runs over leptons e, μ, and τ; ν_ℓ runs over associated neutrinos



$= A_-^\mu$

From

$$\langle VAC|\,A_i^\mu\,|\pi_j\rangle = \frac{i F_\pi \delta_{ij}\, p_\pi^\mu\, e^{ip_\pi\cdot x}}{2(2\pi)^{3/2}\sqrt{2p_\pi^0}}$$

one then gets

$$\Gamma(\pi^+ \to \mu^+ + \nu_\mu) \sim G_{wk}^2 F_\pi^2$$

§5.4 Effective Field Theories

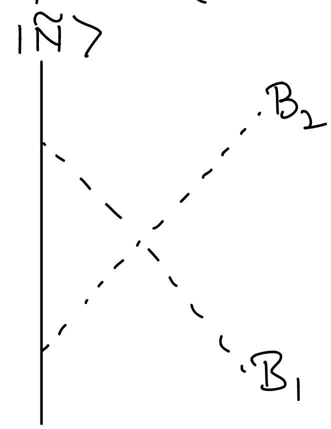
Want to construct current appearing in $\mathcal{L}_{WK}$ from "effective field theories" for pions

⟶ construct Lagrangian that <u>respects</u> the broken symmetry

⟶ construct conserved currents using Noether method

For example, in the emission/absorbtion process of <u>two</u> Goldstone bosons, we must compute matrix elements of the form

$$\langle \beta | T\{ J_1^{\lambda_1}(x_1), J_2^{\lambda_2}(x_2) \} | \alpha \rangle$$



$|N\rangle$ (= $|p\rangle$, $|n\rangle$)
nucleon state

useful for computing the amplitude for the emission of a set of Goldstone bosons:

$\alpha \longrightarrow \beta + B_1 + B_2 + \ldots$

⟶ need effective Lagrangian for Pion-interactions!

σ-model:

start with the $SO(4) = SU(2) \times SU(2)$ invariant Lagrangian:

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi_n \partial^\mu \phi_n - \frac{\mu^2}{2} \phi_n \phi_n - \frac{\lambda}{4} (\phi_n \phi_n)^2, \quad (1)$$

where n is understood to be summed over the values 1, 2, 3, 4, with $SU(2)$-isospin acting as vector-rep. on $\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$ and ϕ_4 an isoscalar

Lagrangian (1) <u>cannot</u> be used for computing scattering amplitudes between Goldstone bosons (no small expansion parameter)

$\longrightarrow$ recast in a way that each Goldstone mode is accompanied by a spacetime derivative

$\longrightarrow$ in Fourier space this becomes the energy (small)

$\rightarrow$ obtain expansion in terms of energy!

Take 4-vector ϕ_n as $(0, 0, 0, \sigma)$ (using rotation matrix R):

$$\phi_n(x) = R_{n4}(x) \sigma(x)$$

with $R^T(x) R(x) = \mathbb{1}$.

Therefore,

$$\sigma(x) = \sqrt{\sum_n \phi_n(x)^2}$$

→ Lagrangian (1) then becomes:

$$\mathcal{L} = -\frac{1}{2}\sum_{n=1}^{4}\left(R_{n4}\partial_\mu\sigma + \sigma\partial_\mu R_{n4}\right)^2 - \frac{1}{2}\mu^2\sigma^2 - \frac{\lambda}{4}\sigma^4$$

Using

$$\sum_n R_{4n}^2 = 1, \quad \sum_n R_{n4}\partial_\mu R_{n4} = \frac{1}{2}\partial_\mu\sum_n R_{n4}^2 = 0,$$

we get

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma - \frac{1}{2}\sigma^2\sum_{n=1}^{4}\partial^\mu R_{n4}\partial_\mu R_{n4} \qquad (2)$$
$$-\frac{1}{2}\mu^2\sigma^2 - \frac{\lambda}{4}\sigma^4$$

For $\mu^2 \leq 0$, σ attains non-vanishing vacuum expectation value : $\bar{\sigma} = |\mu|/\sqrt{\lambda}$

For the remaining fields, choose parametrization:

$$\zeta_a \equiv \frac{\phi_a}{\phi_4 + \sigma}, \quad a = 1, 2, 3 \qquad (*)$$

and take

$$R_{a4} = \frac{2\zeta_a}{1+\vec{\zeta}^2} = -R_{4a}, \quad R_{44} = \frac{1-\vec{\zeta}^2}{1+\vec{\zeta}^2},$$

$$R_{ab} = \delta_{ab} - \frac{2\zeta_a\zeta_b}{1+\vec{\zeta}^2}$$

so that

$$\phi_a/\sigma = R_{a4} = \frac{2\xi_a}{1+\vec{\xi}^2}\ , \quad \phi_4/\sigma = \frac{1-\vec{\xi}^2}{1+\vec{\xi}^2}$$

Then eq. (2) becomes

$$(3) \quad \mathcal{L} = -\frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma - 2\sigma^2\vec{D}_\mu\cdot\vec{D}^\mu - \frac{1}{2}\mu^2\sigma^2 - \frac{\lambda}{4}\sigma^4$$

factor of energy

where $\vec{D}_\mu = \dfrac{\partial_\mu\vec{\xi}}{1+\vec{\xi}^2}$

$\longrightarrow$ fields $\vec{\xi}$ describe particles of zero mass

these are oure new pion fields

$\mathcal{L}$ is invariant under SO(4) which is realized <u>non-linearly</u>:

- under isospin trfs. we have:

$$\delta\vec{\xi} = \vec{\theta}\times\vec{\xi}\ , \qquad \delta\sigma = 0$$

inf. parameter

$\longrightarrow$ $\mathcal{L}$ $SU(2)_{iso}$-invariant

- under broken $SU(2)_{chir}$ we have

$$\delta\vec{\phi} = 2\vec{\varepsilon}\,\phi_4\ , \quad \delta\phi_4 = -2\vec{\varepsilon}\cdot\vec{\phi}$$

from (*) we get:

$$\delta\vec{\xi} = \vec{\varepsilon}(1-\vec{\xi}^2) + 2\vec{\xi}(\vec{\varepsilon}\cdot\vec{\xi}),\quad \delta\sigma = 0$$

$$\rightarrow \delta\vec{D}_\mu = 2(\vec{\xi}\times\vec{\varepsilon})\times\vec{D}_\mu$$

$\rightarrow$ $\mathcal{L}$ remains invariant!